

LIFTING VIA COCYCLE DEFORMATION

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ABSTRACT. We develop a strategy to compute all liftings of a Nichols algebra over a finite dimensional cosemisimple Hopf algebra. We produce them as cocycle deformations of the bosonization of these two. In parallel, we study the shape of any such lifting.

1. INTRODUCTION

Let A be a finite-dimensional Hopf algebra whose coradical is a Hopf subalgebra H . Then the graded algebra associated to the coradical filtration of A is again a Hopf algebra, which is given by a smash product $\text{gr}A \simeq R \# H$, for $R = \bigoplus_{n \geq 0} R^n$ a graded Hopf algebra in ${}^H_H\mathcal{YD}$, the category of Yetter-Drinfeld modules over H . Let $V = R^1$, then the subalgebra of R generated by V is the *Nichols algebra* $\mathcal{B}(V)$; this is a braided Hopf algebra in ${}^H_H\mathcal{YD}$ which is also defined for every $V \in {}^H_H\mathcal{YD}$ by a universal quotient $T(V)/\mathcal{J}(V)$, for $\mathcal{J}(V)$ an ideal generated by homogeneous elements of degree ≥ 2 .

If $\text{gr}A = \mathcal{B}(V) \# H$, then A is called a *lifting* or *deformation* of $\mathcal{B}(V)$ (over H). Hence, deformations of $\mathcal{B}(V)$ give rise to new examples of Hopf algebras. Moreover, there are classes of Hopf algebras (as pointed Hopf algebras over abelian groups) in which every example arises as such a deformation.

1.1. The problem. In this article, we develop a strategy to compute all the liftings or deformations of a Nichols algebra. More precisely, we consider

(1.1) a Hopf algebra H which is finite-dimensional and cosemisimple;

(1.2) $V \in {}^H_H\mathcal{YD}$ such that $\mathcal{J}(V)$ is finitely generated.

The problem is to describe all Hopf algebras A such that

(1.3) $\text{gr}A \simeq \mathcal{B}(V) \# H$.

Notice that the coradical of A is isomorphic to H by (1.3), see [AS2]. This problem is one of the steps in the Lifting Method [AS1, AS2], see also the generalization proposed in [AC]. To deal with it, we split it into two parts:

- (a) To detect the shape of all possible deformations.
- (b) To show that these proposed deformations actually are so.

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Problem (a) is usually taken by examination of the comodule structure of the first term of the coradical filtration, what would give possible deformations by defining relations, see Section 4.

However it is not apparent that the proposed deformations have the desired property; namely, such deformation A would bear an epimorphism $\mathcal{B}(V)\#H \rightarrow \text{gr}A$ but whether this is an isomorphism requires an extra reasoning. This is Problem (b) and there have been different approaches to face up to it: the Diamond Lemma [AS1, AG2, AV1]; a reduction to the first term of the coradical filtration followed by some representation theory, assuming that the Nichols algebra is quadratic [GGI]; a combination of deformation by cocycles and an examination of the PBW basis [AS3].

We briefly recall this last approach highlighting some features that are present in the strategy below; see *loc. cit.* for more details and undefined notation. There, H is assumed to be the group algebra of a finite abelian group Γ (with some restrictions on the order) and $V \in {}^H_H\mathcal{YD}$ has a finite-dimensional Nichols algebra; therefore, by the restrictions alluded to, V is of Cartan type and gives rise to a Dynkin diagram Δ . The defining ideal $\mathcal{J}(V)$ is generated by three kind of relations:

- (i) Serre relations in the same connected component of Δ ,
- (ii) Serre relations between vertices in different connected components,
- (iii) powers of root vectors.

It is then shown that in any deformation A the Serre relations in the same connected component still hold, and the other relations deform respectively to the so-called linking relations, controlled by a parameter λ , and the so-called power of root vector relations, controlled by a parameter μ . Hence the A should be of the form $u(\mathcal{D}, \lambda, \mu) = T(V)\#H/\mathcal{J}$, where the ideal \mathcal{J} is generated by:

- (i) Serre relations (in the same connected component),
- (ii) linking relations,
- (iii) power of root vector relations.

To show that $u(\mathcal{D}, \lambda, \mu)$ has the desired dimension $\dim \mathcal{B}(V)|\Gamma|$, the procedure in [AS3] goes as follows.

- (a) Let $U(\mathcal{D}, \lambda) = T(V)\#H/\mathcal{J}_0$, where the ideal \mathcal{J}_0 is generated by the Serre relations (in the same connected component) and the linking relations. Then $U(\mathcal{D}, \lambda)$ has the “right” basis; it is proved by induction on the number of connected components, via cocycle deformation in the inductive step.
- (b) Finally, $u(\mathcal{D}, \lambda) = U(\mathcal{D}, \lambda)/\mathcal{J}_1$, where \mathcal{J}_1 is generated by the power of root vector relations, has the right dimension by a delicate argument using centrality of these last relations in $U(\mathcal{D}, \lambda)$.

1.2. The background. The family of Hopf algebras $u(\mathcal{D}, \lambda, \mu)$ contains the liftings of quantum linear spaces defined in [AS1]. It was shown in [Ma2] that these liftings of quantum linear spaces are cocycle deformations of their

associated graded Hopf algebras. Further work in this direction was done in [D, BDR, GrM]; in this last paper it was stated that any Hopf algebra $u(\mathcal{D}, \lambda, \mu)$ is a cocycle deformation of its associated graded Hopf algebra, but the argument had a gap and a complete proof was given in [Ma4].

The result in [Ma4] is first extended to the non-abelian case in [GIM] where it is shown that every finite-dimensional pointed Hopf algebra H over \mathbb{S}_3 and \mathbb{S}_4 is again a cocycle deformation of $\text{gr}H$. In [AV2] it is shown that this also the case for finite-dimensional copointed Hopf algebras over \mathbb{S}_3 . Also, in [GIV] this is shown for some pointed or copointed Hopf algebras associated to affine racks. In all of these papers the results are achieved by computing Hopf biGalois objects. In [GM], the authors pick up the work in [GrM] to explicitly compute cocycles as exponentials of Hochschild 2-cocycles. They show that every finite-dimensional pointed Hopf algebra H over the dihedral groups D_{4t} is a cocycle deformation of $\text{gr}H$.

1.3. The strategy. In the present paper, we propose to reverse the order and start by computing all cocycle deformations following ideas in [Ma4]. Observe that, since a deformation by cocycle affects only the multiplication, the coradical filtration of a cocycle deformation A of $\mathcal{B}(V)\#H$ remains unchanged, hence it is isomorphic to $\mathcal{B}(V)\#H$ as coalgebras. Also, it is possible to decide when A is a lifting of $\mathcal{B}(V)$ over H .

Set $\mathcal{T}(V) = T(V)\#H$, $\mathcal{H} = \mathcal{B}(V)\#H$. Our strategy is as follows:

- (a) We decompose a minimal set of generators of the ideal defining $\mathcal{B}(V)$ and recover \mathcal{H} as the last link in a chain of subsequent quotients $\mathcal{T}(V) \twoheadrightarrow \mathcal{H}^1 \twoheadrightarrow \cdots \twoheadrightarrow \mathcal{H}^n \twoheadrightarrow \mathcal{H}$. We choose this decomposition in such a way that every intermediate quotient is achieved by dividing by skew-primitive elements.
- (b) At each step, we compute the Galois objects of \mathcal{H}^{i+1} as quotients of the Galois objects of \mathcal{H}^i , following the results in [Gu]. We start with the trivial Galois object for $\mathcal{T}(V)$. In the final step, we have a set Λ of Galois objects of \mathcal{H} and hence a list of cocycle deformations L , which arise as $L \simeq L(\mathcal{A}, \mathcal{H})$, for $\mathcal{A} \in \Lambda$ as in [S1].
- (c) We check that any lifting is obtained as one of these deformations.

The paper is organized as follows: In Section 2 we fix the notation and introduce the preliminaries on Hopf algebras, Nichols algebras, cocycles and Hopf Galois objects. In Section 3 we recall the two theorems in [Gu] about cleft and Galois objects of quotient Hopf algebras and study the validity of one of the hypothesis of these results in order to apply them in our context. In Section 4 we investigate the shape of any lifting candidate of a Nichols algebra $\mathcal{B}(V)$. We also study this problem in the opposite sense, that is to say we investigate the shape of the graded algebra associated to a deformation. Finally, in Section 5 we present our strategy to compute all cocycle deformations of $\mathcal{B}(V)\#H$. As an illustration, we apply it to classify all liftings of a Nichols algebra associated to an example with diagonal braiding. We end this article with a question related to the extent of this strategy.

2. CONVENTIONS AND PRELIMINARIES

2.1. Conventions. The base field, denoted by \mathbb{k} , is assumed to be algebraically closed and of zero characteristic. Let H be a Hopf algebra. We will use the (summation free) Sweedler's notation for the comultiplication Δ ; ε will denote the counit and \mathcal{S} the antipode. Where needed, we stress the connection with H by a subscript H , *e.g.* Δ_H . We will denote the multiplication by m and the unit by u . We denote by H_0 the *coradical* of H and by $\{H_n\}_{n \in \mathbb{N}}$ the coradical filtration; $G(H)$ is the group of group-like elements and $\widehat{G}(H)$ its group of characters. If A is a right (resp. left) H -comodule algebra, then $A^{\text{co}H}$ (resp. ${}^{\text{co}H}A$) denotes the subalgebra of right (resp. left) coinvariants. If Γ is a group, then $Z(\Gamma)$ denotes its center.

Given a Hopf algebra H with bijective antipode, we denote by ${}^H_H\mathcal{YD}$, resp. \mathcal{YD}_H^H , the category of left, resp. right, Yetter-Drinfeld modules over H . Let $V \in {}^H_H\mathcal{YD}$, $\Gamma = G(H)$. If $g \in Z(\Gamma)$, $\chi \in \widehat{\Gamma}$, then V_χ^g denotes the set of all $x \in V$ with coaction $x \mapsto g \otimes x$ and action $g \cdot x = \chi(g)x$. Also, we denote by $\mathcal{J}(V) = \bigoplus_{n \geq 2} \mathcal{J}^n(V)$ the defining ideal of the Nichols algebra $\mathcal{B}(V)$, *i.e.* $\mathcal{B}(V) = T(V)/\mathcal{J}(V)$. Recall that $\mathcal{B}(V) = \bigoplus_{n \geq 0} \mathcal{B}^n(V)$ is a graded braided Hopf algebra in ${}^H_H\mathcal{YD}$. A pre-Nichols algebra is an intermediate graded braided Hopf algebra between $T(V)$ and $\mathcal{B}(V)$, see [EGNO, Mü] for details about Hopf algebras in braided tensor categories.

2.2. Cocycles. Let H be a Hopf algebra. A 2-cocycle is a convolution-invertible linear map $\sigma : H \otimes H \rightarrow \mathbb{k}$ satisfying, for every $x, y, z \in H$,

$$(2.1) \quad \sigma(x, 1) = \sigma(1, x) = \varepsilon(x) \quad \text{and}$$

$$(2.2) \quad \sigma(x_{(1)}, y_{(1)})\sigma(x_{(2)}y_{(2)}, z) = \sigma(y_{(1)}, z_{(1)})\sigma(x, y_{(2)}z_{(2)}).$$

Let σ be a 2-cocycle. Then $\cdot_\sigma : H \otimes H \rightarrow H$, given by

$$(2.3) \quad x \cdot_\sigma y = \sigma(x_{(1)}, y_{(1)})x_{(2)}y_{(2)}\sigma^{-1}(x_{(3)}, y_{(3)}), \quad x, y, z \in H,$$

defines an associative product on the vector space H with unit u_H . Moreover, the collection $(H, \cdot_\sigma, u_H, \Delta, \varepsilon, \mathcal{S}_\sigma)$ is a Hopf algebra with antipode $\mathcal{S}_\sigma = f * \mathcal{S} * f^{-1}$, for $f = \sigma \circ (\text{id} \otimes \mathcal{S}) \circ \Delta$. This Hopf algebra is denoted H_σ .

The group of convolution-invertible linear functionals of H acts on the set $Z^2(H, \mathbb{k})$ of 2-cocycles. If $\alpha \in \text{Hom}(H, \mathbb{k})$ is convolution-invertible, then

$$\sigma^\alpha(x, y) = \alpha(x_{(1)})\alpha(y_{(1)})\sigma(x_{(2)}, y_{(2)})\alpha^{-1}(x_{(3)}y_{(3)}), \quad \forall x, y \in H$$

is again a 2-cocycle and $\alpha^{-1} * \text{id} * \alpha : H_{\sigma^\alpha} \rightarrow H_\sigma$ is an isomorphism of Hopf algebras. The quotient of $Z^2(H, \mathbb{k})$ under this action is denoted $H^2(H, \mathbb{k})$.

Remarks 2.1. Let H be a Hopf algebra and let $\sigma : H \otimes H \rightarrow \mathbb{k}$ be a 2-cocycle. Since the comultiplications of H and H_σ coincide, we have

- (a) The coradicals of H and H_σ coincide.
- (b) The coradical filtrations of H and H_σ coincide; this is valid for any wedge filtration (*e.g.* the *standard filtration* defined in [AC]).
- (c) If C and D are subcoalgebras of H , then $C \cdot D = C \cdot_\sigma D$.

- (d) Let C be a subcoalgebra stable by the antipode \mathcal{S}_H . Let K be the subalgebra of H generated by C (a Hopf subalgebra indeed) and set $\sigma' = \sigma|_{K \otimes K}$. Then $K_{\sigma'}$ is the subalgebra of $H_{\sigma'}$ generated by C .

Given a 2-cocycle $\sigma : H \otimes H \rightarrow \mathbb{k}$ there is another way to define an associative product on H :

$$(2.4) \quad x \cdot_{(\sigma)} y = \sigma(x_{(1)}, y_{(1)})x_{(2)}y_{(2)}, \quad x, y, z \in H.$$

We denote this algebra by $H_{(\sigma)}$. Then $\Delta : H_{(\sigma)} \rightarrow H_{(\sigma)} \otimes H$ is an algebra map and $H_{(\sigma)}$ becomes a right H -comodule algebra. Moreover, $H_{(\sigma)}^{\text{co}H} = \mathbb{k}$.

This notion extends to any braided tensor category \mathcal{C} . If $R \in \mathcal{C}$ is a braided Hopf algebra, a *braided cocycle* $\sigma \in \text{Hom}(R \otimes R, \mathbf{1})$ is a convolution-invertible map such that

$$(2.1') \quad \sigma(1_R \otimes \text{id}) = \sigma(\text{id} \otimes 1_R) = \varepsilon \quad \text{and}$$

$$(2.2') \quad (\sigma \otimes \varepsilon) * \sigma(m \otimes \text{id}) = (\varepsilon \otimes \sigma) * \sigma(\text{id} \otimes m).$$

If R is a braided Hopf algebra and σ is a braided 2-cocycle, then the coalgebra R with multiplication $m_{\sigma} = \sigma * m * \sigma^{-1}$ is a braided Hopf algebra.

2.3. Galois objects. Let H be a Hopf algebra and let A be a right H -comodule algebra with $\mathbb{k} \simeq A^{\text{co}H}$. Then A is a (right) *H -Galois object* if the *canonical* linear map $\text{can} : A \otimes A \rightarrow A \otimes H$, $a \otimes b \mapsto ab_{(0)} \otimes b_{(1)}$ is an isomorphism. Left H -Galois objects are defined analogously. We set $\text{Gal}(H) = \{\text{isomorphism classes of (right) } H\text{-Galois objects}\}$. Let H, L be Hopf algebras. An (H, L) -bicomodule algebra is an (H, L) -*biGalois object* if it is both a left H -Galois object and a right L -Galois object.

Let A be a right H -comodule algebra with $A^{\text{co}H} = \mathbb{k}$. Then A is a *cleft object* when there is a convolution-invertible comodule map $\gamma : H \rightarrow A$, or equivalently when $A \simeq H_{(\sigma)}$ for some 2-cocycle σ , see [Mo, Section 7] and [DT2]. We may assume that $\gamma(1) = 1$, in which case γ is called a *section*. The cocycle $\sigma : H \otimes H \rightarrow \mathbb{k}$ is given by

$$(2.5) \quad \sigma(h, k) = \gamma(h_{(1)})\gamma(k_{(1)})\gamma^{-1}(h_{(2)}k_{(2)}), \quad h, k \in H.$$

Set $\text{Cleft}(H) := \{\text{isomorphism classes of } H\text{-cleft objects}\} \simeq H^2(H, \mathbb{k})$.

Remarks 2.2. (1) A is cleft if and only if A is an H -Galois object and has the *normal basis* property, *i.e.* $A \simeq H$ as right H -comodules [DT2]. If A is cleft and $\gamma : H \rightarrow A$ is a section, then

$$(2.6) \quad \text{can}^{-1}(a \otimes h) = a\gamma^{-1}(h_{(1)}) \otimes \gamma(h_{(2)}), \quad a \in A, h \in H.$$

- (2) If H is pointed, then any H -Galois object is cleft [Gu, Remark 10].

Given a right H -Galois object A , there is a Hopf algebra $L = L(A, H)$ attached to the pair (A, H) in such a way that A becomes an (L, H) -biGalois object [S1, Section 3]. As a vector space, $L(A, H) = (A \otimes A)^{\text{co}H}$, it is a

subalgebra of $A \otimes A^{\text{op}}$; the coproduct Δ_L and the coaction $\lambda : A \rightarrow L \otimes A$ are, for $\sum_i x_i \otimes y_i \in L$ and $x \in A$:

$$(2.7) \quad \Delta_L \left(\sum_i x_i \otimes y_i \right) = \sum_i x_{i(0)} \otimes \text{can}^{-1}(1 \otimes x_{i(1)}) \otimes y_i,$$

$$(2.8) \quad \lambda(x) = x_{(0)} \otimes \text{can}^{-1}(1 \otimes x_{(1)}).$$

The Hopf algebra L is uniquely characterized by this property [S1, Theorem 3.3]: if L' is another bialgebra and λ' is a left L' -coaction on A making it an (L', H) -biGalois object, then there exists a unique isomorphism $\psi : L \rightarrow L'$ such that $\lambda' = (\psi \otimes \text{id})\lambda$. Explicitly, see [S1, Lemma 3.2],

$$(2.9) \quad \psi \left(\sum_i x_i \otimes y_i \right) \otimes 1_A = \sum_i \lambda'(x_i)(1 \otimes y_i), \quad \sum_i x_i \otimes y_i \in L.$$

Remark 2.3. If $\sigma \in Z^2(H, \mathbb{k})$, then $L(H_{(\sigma)}, H) \simeq H_{\sigma}$ [S1, Theorem 3.9].

3. HOPF GALOIS OBJECTS FOR QUOTIENT HOPF ALGEBRAS

Our argument involves a recurrence on a chain of Hopf algebra quotients. We will use [Gu, Theorems 4 & 8], which we cite next, to study cocycle deformations for a quotient Hopf algebra.

Before stating the theorem, we need some preliminaries. Let L be a Hopf algebra with bijective antipode and let K be a quotient Hopf algebra. Then the right coideal subalgebra $X = {}^{\text{co}K}L$ of the left coinvariants is an algebra in \mathcal{YD}_L^L with coaction given by $\Delta_{L|X}$ and action $x \cdot l = \mathcal{S}(l_{(1)})xl_{(2)}$.

Let $A \in \text{Gal}(L)$ and set $\sum_i \ell_i(h) \otimes r_i(h) = \text{can}^{-1}(1 \otimes h) \in A \otimes A$. Then A is an algebra in \mathcal{YD}_L^L via the *Miyashita-Ulbrich action* $a \cdot h = \sum_i \ell_i(h)ar_i(h)$, $a \in A, h \in L$, see [DT1]. If A, B are algebras in \mathcal{YD}_L^L , $\text{Alg}_L^L(A, B)$ denotes the set of algebra morphisms between them.

Theorem 3.1. [Gu, Theorem 4] *Let L , K and $X = {}^{\text{co}K}L$ be as above. Assume that L is left and right K -cflat. There are bijective correspondences*

$$\text{Gal}(K) \xrightleftharpoons[\Psi]{\Phi} \{(A, f) : [A] \in \text{Gal}(L), f \in \text{Alg}_L^L(X, A)\} / \sim,$$

$$\Psi([(A, f)]) = [A/Af(X^+)], \quad \Phi([B]) = [(B \square_K L, x \mapsto 1 \otimes x)].$$

If there is a subcoalgebra of L that is mapped isomorphically onto the coradical of K , then this correspondence restricts to cleft objects. \square

In the theorem above, if $\pi : L \rightarrow K$ is the projection, the coaction on $A/Af(X^+)$ is given by $(\tau \otimes \pi)\lambda_A$, for $\tau : A \rightarrow A/Af(X^+)$ the projection. If $B \in \text{Gal}(K)$, then the L -coaction on $B \square_K L$ is $\text{id}_B \otimes \Delta_L$. In turn, the equivalence \sim is defined so that $(A, f) \sim (A', f')$ if and only if there exists an isomorphism $\alpha : A \rightarrow A'$ of L -comodule algebras such that $f' = \alpha \circ f$.

In order to apply Theorem 3.1 we need to investigate when a Hopf algebra L is coflat over a quotient Hopf algebra K . This will be the content of Subsection 3.1. Besides, we need to compute the algebra of coinvariants ${}^{\text{co}K}L$; this is not always easy. There is another approach to computing cleft objects of quotient Hopf algebras given by [Gu, Theorem 8], which we recall next. See Section 5.6 for a detailed comparison of these two theorems.

If A, A' are right L -comodule algebras, $\text{Alg}^L(A, A')$ is the set of comodule algebra morphisms between them. If $X \subset L$ is a right coideal subalgebra, $N(X)$ is the subalgebra generated by $\{\mathcal{S}(h_{(1)})xh_{(2)} : h \in L, x \in X\}$.

Theorem 3.2. [Gu, Theorem 8] *Let L be a pointed Hopf algebra and let $X \subset L$ be a right coideal subalgebra, $X^+ = \ker \varepsilon|_X$. Set $I = LX^+L$, $K = L/I$; then K is a quotient Hopf algebra of L . Assume that ${}^{\text{co}K}L \cap L_0 \subseteq N(X)$. Then there are bijective correspondences*

$$\text{Cleft}(K) \xrightleftharpoons[\Psi]{\Phi} \left\{ (A, f) : \begin{array}{l} [A] \in \text{Cleft}(L), f \in \text{Alg}^L(X, A) \\ \text{such that } Af(X^+)A \neq A \end{array} \right\} / \sim,$$

$$\Psi([(A, f)]) = [A/Af(X^+)A], \quad \Psi([B]) = [(B \square_K L, x \mapsto 1 \otimes x)].$$

The corresponding coactions and the relation \sim are as in Theorem 3.1. \square

Remark 3.3. In many of the cases we shall study, we have $N(X) = {}^{\text{co}K}L$, which necessarily ensures the hypothesis ${}^{\text{co}K}L \cap L_0 \subseteq N(X)$ above. Also, this equality allows us to drop the hypothesis of L being pointed, as this assumption is used in [Gu] precisely to see that $N(X) = {}^{\text{co}K}L$.

3.1. On the coflatness of quotients. Fix a Hopf algebra H with bijective antipode. Let R be a connected (*i.e.* the coradical of R is \mathbb{k}) Hopf algebra in ${}^H_H\mathcal{VD}$. In particular, the antipode of R , hence that of $A = R \# H$, is bijective. Clearly, see [Ma3, Section 1], we have

- If B is a right coideal subalgebra of R , then R/RB^+ is a quotient left R -module coalgebra of R .
- If T is a quotient left R -module coalgebra of R , then the left T -coinvariants ${}^{\text{co}T}R$ form a right coideal subalgebra of R .

Recall that the structures on R arise from the obvious Hopf algebra maps $H \rightarrow A \rightarrow H$, whose composite is the identity on H , as follows: we have $R = A^{\text{co}H}$, so that R is a left coideal subalgebra of A , and is thus an algebra and left H -comodule, while we have $R = A/AH^+$, so that R is a quotient left A -module coalgebra of A , and is thus a coalgebra and left H -module; the last left H -module structure coincides with the adjoint action. Let P (resp. Q) denote the braided Hopf algebra in ${}^{H^{\text{op}}}_{H^{\text{op}}}\mathcal{VD}$ (resp. ${}^{H^{\text{cop}}}_{H^{\text{cop}}}\mathcal{VD}$) which arises from $H^{\text{op}} \rightarrow A^{\text{op}} \rightarrow H^{\text{op}}$ (resp. $H^{\text{cop}} \rightarrow A^{\text{cop}} \rightarrow H^{\text{cop}}$). Then $P = Q = R$ as vector spaces. As an algebra, P equals the opposite algebra R^{op} of R , while as a coalgebra, Q equals the co-opposite coalgebra R^{cop} of R .

Lemma 3.4. (i) *The sub-objects (resp., right coideal subalgebras) of R in ${}^H_H\mathcal{YD}$ coincide with those of P in ${}^{H^{\text{op}}}_{H^{\text{op}}}\mathcal{YD}$.*

(ii) *The quotient objects (resp., left module coalgebras) of R in ${}^H_H\mathcal{YD}$ coincide with those of Q in ${}^{H^{\text{cop}}}_{H^{\text{cop}}}\mathcal{YD}$.*

Proof. (i) Since the comultiplication does not change, $R = P$ as left comodules over the coalgebra $H = H^{\text{op}}$. If $h = \mathcal{S}_H^{-1}(k) \in H$, $a \in A$, then the adjoint actions of H and H^{op} are related by

$$\text{ad}_H h(a) = h_{(1)} a \mathcal{S}_H(h_{(2)}) = k_{(1)} \cdot_{\text{op}} a \cdot_{\text{op}} \mathcal{S}_{H^{\text{op}}}(k_{(2)}) = \text{ad}_{H^{\text{op}}} k(a).$$

This settles the claim for sub-objects. Let now X be a sub-object of R , or equivalently of P . Clearly, X is a subalgebra of R if and only if it is a subalgebra of $P = R^{\text{op}}$. For $x \in R$, let $x \mapsto \sum x^{(1)} \otimes x^{(2)}$ denote the coproduct on R . Then the coproduct $\Delta(x)$ on A is given by

$$\Delta(x) = (x^{(2)})_{(-1)} (\mathcal{S}_H^{-1}((x^{(2)})_{(-2)})) \rightharpoonup x^{(1)} \otimes (x^{(2)})_{(0)}.$$

Hence Δ_P is given by $x \mapsto \mathcal{S}_H^{-1}((x^{(2)})_{(-1)}) \rightharpoonup x^{(1)} \otimes (x^{(2)})_{(0)}$. It follows that X is a right coideal of R , if and only if it is such of P . (ii) is similar. \square

Let B be a right coideal subalgebra of R in ${}^H_H\mathcal{YD}$. Then one can define the category $({}^H_H\mathcal{YD})_B^R$ of right (R, B) -Hopf modules in ${}^H_H\mathcal{YD}$; a *right (R, B) -Hopf module* is here a right B -module and right R -comodule in ${}^H_H\mathcal{YD}$ which satisfies the compatibility condition formulated as in the ordinary situation, but involving the braiding $R \otimes B \xrightarrow{\sim} B \otimes R$.

Lemma 3.5. *Every object M in $({}^H_H\mathcal{YD})_B^R$ includes a sub-object X in ${}^H_H\mathcal{YD}$ such that the action map $X \otimes B \rightarrow M$ is a bijection, necessarily an isomorphism in $({}^H_H\mathcal{YD})_B$.*

Proof. The lemma follows from the following claim.

Claim. If $M \neq 0$, then M includes a non-zero sub-object N in $({}^H_H\mathcal{YD})_B^R$ which includes a sub-object X in ${}^H_H\mathcal{YD}$ such that $X \otimes B \xrightarrow{\sim} N$.

Indeed, assume that we have proven the claim. We consider all pairs (N, X) , where N is a sub-object of M in $({}^H_H\mathcal{YD})_B^R$, and X is a sub-object of N in ${}^H_H\mathcal{YD}$ such that $X \otimes B \xrightarrow{\sim} N$ naturally, and introduce the natural order given by inclusion to the pairs. By Zorn's Lemma we have a maximal pair (N, X) . To see $N = M$, suppose on the contrary $N \subsetneq M$. With the assumed fact applied to M/L , we have sub-objects $\tilde{N} \subset M$ in $({}^H_H\mathcal{YD})_B^R$ and $\tilde{X} \subset \tilde{N}$ in ${}^H_H\mathcal{YD}$ such that $N \subsetneq \tilde{N}$, $X \subsetneq \tilde{X}$ and $\tilde{X}/X \otimes B \xrightarrow{\sim} \tilde{N}/N$. A map of short exact sequences which is isomorphic on the kernels and the cokernels shows that $\tilde{X} \otimes B \xrightarrow{\sim} \tilde{N}$, which contradicts the maximality of (N, X) , and hence shows $N = M$. This argument is the same as the one in [Ra, Proposition 1].

We now prove the claim. Suppose $0 \neq M \in ({}^H_H\mathcal{YD})_B^R$. We wish to prove M includes a nonzero pair. Set $X = M^{\text{co}R}$. This is a subject of M in ${}^H_H\mathcal{YD}$,

and is the socle $\text{soc } M$ of the right R -comodule M , whence $X \neq 0$. The tensor product $X \otimes B$ is naturally an object in $({}^H_H\mathcal{YD})_B^R$ whose R -comodule socle $\text{soc}(X \otimes B) = X$. We see that $f : X \otimes B \rightarrow M$, $f(v \otimes b) = vb$ is a morphism in $({}^H_H\mathcal{YD})_B^R$, which is injective since it is restricted to the identity on the socles. If $L = \text{Im } f = XB$ then (N, X) is a desired pair. \square

Proposition 3.6. *Let R be a connected Hopf algebra in ${}^H_H\mathcal{YD}$.*

- (a) *R is a free left and right module over every right coideal subalgebra.*
- (b) *R is a cofree left and right comodule over every quotient left module coalgebra T .*
- (c) *$B \mapsto R/RB^+$ and $T \mapsto {}^{\text{co}T}R$ give a bijection between the set of right coideal subalgebras B of R and the set of quotient left R -module coalgebras T of R .*
- (d) *If B and T correspond to each other via this bijection, then there exists a left T -colinear and right B -linear isomorphism $T \otimes B \xrightarrow{\cong} R$.*

Proof. (a) Let B be a right coideal subalgebra. First, we prove the right B -freeness. Notice that $R \in ({}^H_H\mathcal{YD})_B^R$. The right B -freeness in (a) follows from Lemma 3.5. By Lemma 3.4 (i), the just proved result applied to the P of the lemma implies the left B -freeness¹.

(c) Let T be as in (b), and set $B = {}^{\text{co}T}R$. We see that the natural left T -comodule structure $R \rightarrow T \otimes R$ on R is right B -linear. The base extension along $B \rightarrow R$ induces a right B -linear and left T -colinear map $g : R \otimes_B R \rightarrow T \otimes R$. This is induced from the canonical isomorphism $R \otimes R \xrightarrow{\cong} R \otimes R$, and hence is a surjection. Note that T is also connected.

Since R is left B -free as was shown in (a), the left T -comodule socle of $R \otimes_B R$ equals $B \otimes_B R$. It follows that g is injective, and hence bijective, since it restricts to id_R on the left T -comodule socles.

The bijection g together with the left B -freeness of R shows that R is an injective cogenerator (or equivalently, faithfully coflat) as a left T -comodule; see also [Ma3, Proposition 1.4(1)]. The desired one-to-one correspondence follows just as in the ordinary situation; see [Ma3, Proposition 1.4(2)].

(d) Let B and T correspond to each other. The left T -injectivity of R allows the inclusion $k \rightarrow R$ of left T -comodules to extend to a unit-preserving left T -colinear map $h : T \rightarrow R$. The right B -linearization of h

$$h_B : T \otimes B \rightarrow R, \quad h_B(t \otimes b) = h(t)b$$

is right B -linear and left T -colinear and injective, since it restricts to id_B on the T -comodule socles. It is an isomorphism, since $T \otimes B$ is T -injective.

(b) Let T be as in (b). We see from Part (d) that R is left T -cofree. Lemma 3.4 (ii) shows that R is also right T -cofree. \square

¹One can define the analogous category ${}_B({}^H_H\mathcal{YD})^R$. But, it is impossible to discuss as above, because for $M \in {}_B({}^H_H\mathcal{YD})^R$, the right R -comodule $B \otimes M^{\text{co}R}$ is not isomorphic to a direct sum of copies of B , and so $\text{soc}(B \otimes M^{\text{co}R}) = M^{\text{co}R}$ may not be true.

Corollary 3.7. *Let H be a cosemisimple Hopf algebra, let R, T be connected braided Hopf algebras in ${}^H_H\mathcal{YD}$, such that T is a quotient of R . Then $R\#H$ is left and right cofree over $T\#H$. In particular, it is left and right coflat.*

Proof. As H is cosemisimple, the coalgebra surjection $\text{id} \otimes \varepsilon : T\#H \rightarrow T$ is a cosemisimple coextension, that is a left or right $T\#H$ -comodule is injective if it is injective as an T -comodule. Since R is left T -cofree as a T -comodule and so left T -injective, it follows that $R\#H$, being left T -injective, is left $T\#H$ injective. Note that the coradical of $T\#H$ is isomorphically liftable to the coradical of $R\#H$, since both of them coincide with H . It follows by [Ma1] that there is a left $T\#H$ -colinear and right ${}^{\text{co}T\#H}(R\#H)$ -linear isomorphism

$$R\#H \simeq T\#H \otimes {}^{\text{co}T\#H}(R\#H).$$

By switching the sides one can present $R\#H, T\#H$ as smash products $H\#R', H\#T'$ of braided Hopf algebras R', T' in \mathcal{YD}_H^H , such that T' is a quotient of R' , and prove that R' is right (and left) T' -injective, which shows as above that there is a right $T\#H$ -colinear and left $(R\#H)^{\text{co}T\#H}$ -linear isomorphism $R\#H \simeq (R\#H)^{\text{co}T\#H} \otimes (T\#H)$. \square

4. THE SHAPE OF ALL POSSIBLE DEFORMATIONS

Fix H as in (1.1) and A a Hopf algebra whose coradical is isomorphic to H . By [AMS, Theorem 5.9.c)] there exists a coalgebra H -bimodule projection $\Pi : A \rightarrow H$ such that $\Pi|_H = \text{id}_H$. Hence A is a Hopf bimodule coalgebra over H via the left and right multiplication and the coactions $\rho_L = (\Pi \otimes \text{id})\Delta$ and $\rho_R = (\text{id} \otimes \Pi)\Delta$. Let $P_0 = 0, P_1 = \{x \in A : \Delta(x) = \rho_L(x) + \rho_R(x)\}$ and

$$P_n = \{x \in A : \Delta(x) - \rho_L(x) - \rho_R(x) \in \sum_{i=1}^{n-1} P_i \otimes P_{n-i}\}.$$

Then $P_n = A_n \cap \ker \Pi$ [AN, Lemma 1.1]. Clearly, P_n is a Hopf sub-bimodule of A_n and $A_n/A_{n-1} = P_n/P_{n-1}$. The canonical projection $\pi_n : A_n \rightarrow A_n/A_{n-1}$ is a Hopf bimodule map. As H is semisimple there exists a section ι_n of π_n . Hence $A = H \oplus \bigoplus_{n \geq 1} \iota_n(P_n/P_{n-1})$. We extend π_n to be 0 in $\bigoplus_{m > n} \iota_m(P_m/P_{m-1})$ if $n > 0$ and set $\pi_0 = \Pi$. We shall generally omit ι_m .

We recall the structure of $\text{gr}A$. As vector spaces, $\text{gr}A(n) = A_n/A_{n-1} = P_n/P_{n-1}$. The multiplication and comultiplication of $\text{gr}A$ are

$$\begin{aligned} \pi_n(x)\pi_m(y) &= \pi_{n+m}(xy), \quad x \in A_n, y \in A_m \text{ and} \\ \Delta_{\text{gr}A}(\pi_n(x)) &= \sum_{i=0}^n \pi_i(x_{(1)}) \otimes \pi_{n-i}(x_{(2)}), \quad x \in A_n. \end{aligned}$$

By abuse of notation, π_0 denotes the projection $\text{gr}A \twoheadrightarrow H$ with kernel $\bigoplus_{n > 0} \text{gr}A(n)$ and $\pi_0|_H = \text{id}$. Then $\text{gr}A$ is a Hopf bimodule over H via the left and right multiplication and the coactions $(\pi_0 \otimes \text{id})\Delta_{\text{gr}A}$ and $(\text{id} \otimes \pi_0)\Delta_{\text{gr}A}$.

It is well known that $\text{gr}A \simeq (\text{gr}A)^{\text{co}H} \# H$ as Hopf algebras. In [AMS, Theorem 5.23], it is shown that $A^{\text{co}H}$ is a coalgebra in ${}^H_H\mathcal{YD}$ such that $A \simeq A^{\text{co}H} \# H$ as coalgebras and the multiplication in A is recovered with an extra structure on $A^{\text{co}H}$, see also [S2]. Note that if $\dim A < \infty$ and $A \simeq \text{gr}A$ as coalgebras, or equivalently $A^{\text{co}H} \simeq (\text{gr}A)^{\text{co}H}$ as coalgebras in ${}^H_H\mathcal{YD}$, then A is a cocycle deformation of $\text{gr}A$ [S1, Corollary 5.9].

Lemma 4.1. $\pi_n : \iota_n(P_n/P_{n-1}) \rightarrow \text{gr}A(n)$ is an isomorphism of Hopf bimodules over H for all n . Therefore $A^{\text{co}H} \simeq (\text{gr}A)^{\text{co}H}$ in ${}^H_H\mathcal{YD}$.

Proof. If $x \in \iota_n(P_n/P_{n-1})$ and $h \in H$, then $\pi_n(hx) = \pi_0(h)\pi_n(x)$ and $\pi_n(xh) = \pi_n(x)\pi_0(h)$. Also,

$$\begin{aligned} (\text{id} \otimes \pi_0)\Delta^{\text{gr}}(\pi_n(x)) &= \sum_{i=0}^n \pi_i(x_{(1)}) \otimes \pi_0 \circ \pi_{n-i}(x_{(2)}) = \pi_n(x_{(1)}) \otimes \pi_0(x_{(2)}) \\ &= (\pi_n \otimes \text{id})(\text{id} \otimes \Pi)\Delta(x) = (\pi_n \otimes \text{id})\rho_R(x). \end{aligned}$$

Analogously, π_n is a left comodule map. The last assertion is easy. \square

We fix $V \in {}^H_H\mathcal{YD}$. Set $\mathcal{B}(V) = T(V)/\mathcal{J}(V)$ and $\mathcal{T}(V) = T(V) \# H$.

Definition 4.2. A *lifting map* is an epimorphism $\phi : \mathcal{T}(V) \rightarrow A$ of Hopf algebras such that $\phi|_H = \text{id}_H$ and $\phi|_{V \# H} : V \# H \rightarrow P_1$ is an isomorphism of Hopf bimodules over H .

Proposition 4.3. [AV1, Prop. 2.4] *Let A be a Hopf algebra whose coradical is a Hopf subalgebra isomorphic to H . Then A is a lifting of $\mathcal{B}(V)$ over H if and only if there exists a lifting map $\phi : \mathcal{T}(V) \rightarrow A$.* \square

From now on, we assume that A is a lifting of $\mathcal{B}(V)$ over H with lifting map $\phi : \mathcal{T}(V) \rightarrow A$. Let $\mathbb{B}_{\mathcal{J}}^n$ be a basis of $\mathcal{J}^n(V)$ and extend it to a basis $\mathbb{B}^n \cup \mathbb{B}_{\mathcal{J}}^n$ of $V^{\otimes n}$. We still denote by \mathbb{B}^n the basis of the quotient $\mathcal{B}^n(V)$. Then $\mathbb{B} = \bigcup_n \mathbb{B}^n$ is a basis of $\mathcal{B}(V)$, $\mathbb{B}_{\mathcal{J}} = \bigcup_n \mathbb{B}_{\mathcal{J}}^n$ is a basis of $\mathcal{J}(V)$. Let \mathbb{B}_H be a basis of H .

Remarks 4.4. By Lemma 4.1 we have that:

- (a) $\{\phi(x) - \Pi(\phi(x)) : x \in \mathbb{B}_{\mathcal{J}}^n\} \subset P_{n-1}$.
- (b) $\{\phi(x)h - \Pi(\phi(x))h : x \in \mathbb{B}^i, h \in \mathbb{B}_H, 0 < i \leq n\}$ is a basis of P_n .
- (c) $\phi(\mathbb{B}^2)H = \iota_2(P_2/P_1)$ and $A_2 \simeq (\mathcal{B}(V) \# H)_2$ as coalgebras.
- (d) $\{\phi(x)h : x \in \mathbb{B}, h \in \mathbb{B}_H\}$ is a basis of A . Let $\iota : A \rightarrow \mathcal{T}(V)$ be the linear map identifying this basis of A with $\mathbb{B} \# H$.

The shape of the liftings is given by the following proposition.

Proposition 4.5. *Let $\mathcal{I}_A = \langle \{r_i - \iota\phi(r_i)\}_{i \in I} \rangle$, where $\{r_i\}_{i \in I}$ is a set of homogeneous generators of $\mathcal{J}(V)$. If \mathcal{I}_A is a Hopf ideal, then $A = \mathcal{T}(V)/\mathcal{I}_A$.*

Proof. Since $\mathcal{I}_A \cap (\mathbb{K} \oplus V) \# H = 0$, the coradical of $A' := \mathcal{T}(V)/\mathcal{I}_A$ is H by [Mo, Corollary 5.3.5]. Then $\text{gr}(A') \simeq R \# H$ where $R \simeq T(V)/J$ for a Hopf ideal $J \subseteq \mathcal{J}(V)$. Clearly $\{r_i\}_{i \in I} \subset J$, cf. Remark 4.4 (a), then $J = \mathcal{J}(V)$ and $\dim(A'_n/A'_{n-1}) = \dim(A_n/A_{n-1}) \forall n$, hence the proposition follows. \square

If there are no ambiguities, we identify $(\mathbb{k} \oplus V) \# H$ with its image by ϕ omitting the map ι . We explore a case where the hypothesis of Proposition 4.5 is satisfied. Let M be a Yetter-Drinfeld submodule of $T(V)$.

Definition 4.6. M is *compatible* with ϕ if

$$\Delta(\phi(m)) = \phi(m) \otimes 1 + m_{(-1)} \otimes \phi(m_{(0)}) \text{ for all } m \in M.$$

Assume $M \subset T(V)$ is compatible with ϕ . For $m \in M$, we may see $\phi(m)$ as an element of $(\mathbb{k} \oplus V) \# H$. Set a basis $\{m_i\}_{i=1}^r$ of M and let $\{c_{ij}\}_{i,j} \subset H$ be the set of *comatrix elements* associated to M and $\{m_i\}_{i=1}^r$, that is

$$(4.1) \quad (m_i)_{(-1)} \otimes (m_i)_{(0)} = \sum_j c_{ij} \otimes m_j, \quad 1 \leq i \leq r.$$

Next lemma helps us to describe the image $\phi(M)$.

Lemma 4.7. *Let $M \subset T(V)$ be compatible with ϕ . Then*

- (a) $\pi_1 \circ \phi|_M : M \rightarrow V$ is a morphism in ${}^H_H \mathcal{YD}$.
- (b) Assume that M is simple and $V \simeq M^m \oplus P$ with m maximum. Then there exist $\lambda_1, \dots, \lambda_m \in \mathbb{k}$ such that

$$\pi_1 \circ \phi|_M \simeq \lambda_1 \text{id}_M \oplus \dots \oplus \lambda_m \text{id}_M \oplus 0.$$

- (c) For $\{m_i\}_{i=1}^r, \{c_{ij}\}_{i,j}$ as in (4.1) there exist $a_1, \dots, a_r \in \mathbb{k}$ such that

$$(\pi_0 \circ \phi)(m_i) = a_i - \sum_{j=1}^r a_j c_{ij} \quad i = 1, \dots, r.$$

- (d) Let $\Theta : A \rightarrow A'$ be an isomorphism of Hopf algebras and let $\phi' : \mathcal{T}(V) \rightarrow A'$ be a lifting map. If there is no $v \in V$ such that $h \cdot v = \varepsilon(h)v$ for all $h \in H$, then $\Theta\phi(V) = \phi'(V)$.

Proof. (a) Clearly $\phi(M) \subset A_1$. Since $\pi_1 \circ \phi|_M$ is a morphism of bicomodules over H , $(\pi_1 \circ \phi)(M) \subset V$. (b) is a particular case of (a). We prove (c). If M is a simple H -comodule, then (c) follows from [AV1, Lemma 2.1] since

$$\Delta((\pi_0 \circ \phi)(m_i)) = (\pi_0 \circ \phi)(m_i) \otimes 1 + \sum_j c_{ij} \otimes (\pi_0 \circ \phi)(m_j).$$

Assume otherwise that $M = \bigoplus_{l=1}^n M_l$ where each M_l is a simple H -comodule with basis $\{m_i^l\}_i, l = 1, \dots, n$. Then (c) follows as above for the basis $\{m_i^l\}_{i,l}$ of M . Thus (c) follows since ${}^H_H \mathcal{YD}$ is a semisimple category.

(d) If we consider A_1 as a right H -comodule via the projection $(\varepsilon \# 1) \circ \Theta$, then $\Theta\phi(V) \subset (\mathbb{k} \oplus \phi'(V)) \# 1$. Now if we consider A'_1 as a left H -module via $\text{ad} \circ \Theta$, then (d) follows by hypothesis. \square

Lemma 4.7 has been refined in the copointed case, *i.e.* when H is the function algebra on a finite group, in [GIV, Lemma 3.1].

Lemma 4.8. *Let $M = \bigoplus_{i=1}^t M^{n_i} \subset T(V)$ be a graded Yetter-Drinfeld submodule with $M^{n_i} \neq 0$ and $n_i < n_{i+1}$ for all i . Assume that $\mathcal{J}(V) = \langle M \rangle$.*

- (a) M^{n_1} is compatible with ϕ and $I_1 = \langle m - \phi(m) \rangle_{m \in M^{n_1}}$ is a Hopf ideal.
- (b) Let $1 \leq k \leq t$ be such that M^{n_1}, \dots, M^{n_k} are compatible with ϕ and such that $I_k = \langle m - \phi(m) \rangle_{m \in \bigoplus_{i=s}^k M^{n_i}}$ is a Hopf ideal. Assume that

$$(4.2) \quad \Delta(m) - m \otimes 1 - m_{(-1)} \otimes m_{(0)} \in I_k \otimes \mathcal{T}(V) + \mathcal{T}(V) \otimes I_k$$

holds for every $m \in M^{n_{k+1}}$. Then $M^{n_{k+1}}$ is compatible with ϕ and $I_{k+1} = \langle m - \phi(m) \rangle_{m \in \bigoplus_{i=s}^{k+1} M^{n_i}}$ is a Hopf ideal.

Proof. As M generates $\mathcal{J}(V)$, n_1 is the minimum degree of $\mathcal{J}(V)$ and thus M^{n_1} is compatible with ϕ . Using Lemma 4.7 (a) and (c), we see that I_1 is a Hopf ideal. We now proceed by induction. Assume that our claim holds for k . Since $I_k \subset \ker \phi$, $M^{n_{k+1}}$ is compatible with ϕ by (4.2). Moreover, applying Lemma 4.7 (a) and (c), we see that $\langle m - \phi(m) \rangle_{m \in M^{n_{k+1}}}$ is a Hopf ideal in $\mathcal{T}(V)/I_k$ by (4.2). Hence I_{k+1} is a Hopf ideal of $\mathcal{T}(V)$. \square

Next, we characterize the liftings of $\mathcal{B}(V)$ in the setting of Lemma 4.8.

Theorem 4.9. *Let M be as in Lemma 4.8. Assume (4.2) holds for every $1 \leq k \leq t$. Then $A \simeq \mathcal{T}(V)/\mathcal{I}_A$.*

Proof. Follows from Proposition 4.5 and Lemma 4.8 since $\mathcal{I}_A = I_t$. \square

Theorem 4.9 characterizes the liftings in the case in which the relations are deformed by elements in the first term of the coradical filtration. This is the case in [AS3, AG2, AV1, GGI, FG]. However, there exist examples in which this does not hold, see Example 4.10 below, also [He, GIV]. We believe this is also the case of the liftings we classify in Theorem 5.11, for which we do not have an explicit presentation, since the complexity of the computations that requires exceeds our computational resources.

Example 4.10. [GIV, Theorem 5.4] Let \mathbb{F}_5 denote the finite field of 5 elements. The Nichols algebra $\mathcal{B}(\mathbb{F}_5, 2)$ associated to the *affine rack* $(\mathbb{F}_5, 2)$ and constant cocycle $q \equiv -1$, computed in [AG1], has dimension 1280 and it can be presented by generators x_0, \dots, x_4 and relations

$$(4.3) \quad \begin{aligned} & x_i^2, \quad x_i x_j + x_{2j-i} x_i + x_{3i-2j} x_{2j-i} + x_j x_{3i-2j} \quad 0 \leq i, j \leq 4, \\ & x_1 x_0 x_1 x_0 + x_0 x_1 x_0 x_1. \end{aligned}$$

Let C_8 be the cyclic group of order 8 and let t denote a generator. Consider C_8 acting on \mathbb{Z}_5 by $t \cdot i = 2i$, $i \in \mathbb{Z}_5$, and let $\Gamma = \mathbb{Z}_5 \rtimes_2 C_8$. Then $\mathcal{B}(\mathbb{F}_5, 2)$ can be realized in ${}_{\mathbb{k}\Gamma}^{\mathbb{k}\Gamma} \mathcal{YD}$. Let $\mathcal{H} = \mathcal{B}(\mathbb{F}_5, 2) \# \mathbb{k}\Gamma$. Set $g_i = i \times t \in \Gamma$, $i \in \mathbb{Z}_5$. If L is a deformation of \mathcal{H} then there exist scalars $\lambda_1, \lambda_2, \lambda_3 \in \mathbb{k}$ such that L is the quotient of $T(V) \# \mathbb{k}\Gamma$ by the ideal generated by

$$\begin{aligned} & x_0^2 - \lambda_1(1 - g_0^2), \quad x_0 x_1 + x_2 x_0 + x_3 x_2 + x_1 x_3 - \lambda_2(1 - g_0 g_1), \\ & x_1 x_0 x_1 x_0 + x_0 x_1 x_0 x_1 - s_X - \lambda_3(1 - g_0^2 g_1 g_2), \end{aligned}$$

for $s_X = \lambda_2(x_1 x_0 + x_0 x_1) + \lambda_1 g_1^2(x_3 x_0 + x_2 x_3) - \lambda_1 g_0^2(x_2 x_4 + x_1 x_2) + \lambda_2 \lambda_1 g_0^2(1 - g_1 g_2)$. In particular, the defining relation $x_1 x_0 x_1 x_0 + x_0 x_1 x_0 x_1$ of $\mathcal{B}(\mathbb{F}_5, 2)$ is not deformed by group-like or skew-primitive elements in L .

4.1. The shape of the Hopf algebra $L(-, -)$. Let H, V be as in (1.1) and (1.2). As above, we commonly deal with lifting candidates that are presented as quotients of $\mathcal{T}(V)$. Thus, we also need such a presentation for the Hopf algebra $L(-, -)$.

Let $R \in {}^H_H\mathcal{YD}$ be a pre-Nichols algebra over $V \in {}^H_H\mathcal{YD}$. Set $\mathcal{H} = R\#H$ and $\pi : \mathcal{T}(V) \rightarrow \mathcal{H}$ the canonical projection. Consider $\mathcal{T}(V)$ as an \mathcal{H} -comodule algebra via $(\text{id} \otimes \pi)\Delta_{\mathcal{T}(V)}$. Let $A \in \text{Gal}(\mathcal{H})$ such that there is an algebra projection $\tau : \mathcal{T}(V) \rightarrow A$.

Proposition 4.11. *Let \mathcal{H}, A be as above. Assume that $\tau : \mathcal{T}(V) \rightarrow A$ is a comodule morphism. Let $\wp = (\tau \otimes \tau)(\text{id} \otimes \mathcal{S})\Delta_{\mathcal{T}(V)} \in \text{Alg}(\mathcal{T}(V), A \otimes A^{\text{op}})$. Then $\wp(\mathcal{T}(V)) = L(A, \mathcal{H})$.*

Proof. We have the commutative diagram

$$\begin{array}{ccc} L(\mathcal{T}(V), \mathcal{T}(V)) & \hookrightarrow & \mathcal{T}(V) \otimes \mathcal{T}(V) \\ \downarrow & & \downarrow \tau \otimes \tau \\ L(A, \mathcal{H}) & \hookrightarrow & A \otimes A, \end{array}$$

which gives rise to

$$\begin{array}{ccc} L(\mathcal{T}(V), \mathcal{T}(V)) \otimes \mathcal{T}(V) & \xrightarrow{\text{can}^{-1}} & \mathcal{T}(V) \otimes \mathcal{T}(V) \\ \downarrow & & \downarrow \tau \otimes \tau \\ L(A, \mathcal{H}) \otimes A & \xrightarrow{\text{can}^{-1}} & A \otimes A. \end{array}$$

Hence $\tau \otimes \tau$ restricts to a surjection $L(\mathcal{T}(V), \mathcal{T}(V)) \twoheadrightarrow L(A, \mathcal{H})$. Since $\mathcal{T}(V) \simeq L(\mathcal{T}(V), \mathcal{T}(V))$ via $(\text{id} \otimes \mathcal{S})\Delta_{\mathcal{T}(V)}$, the proposition follows. \square

Remark 4.12. (1) In the setting of Theorem 3.1, let $L = \mathcal{T}(V)$, $A = \mathcal{T}(V)$ with the coaction ρ given by comultiplication. Let $\mathcal{H}, \pi : \mathcal{T}(V) \rightarrow \mathcal{H}$ be as above. Set $X = {}^{\text{co}\mathcal{H}}\mathcal{T}(V)$, $f \in \text{Alg}_L^L(X, A)$, and $\overline{A} = A/Af(X)^+$. Let $\tau : A \rightarrow \overline{A}$ be the projection. The coaction $\overline{\rho}$ in \overline{A} is given by $(\tau \otimes \pi)\rho$, that is $\overline{\rho} \circ \tau = (\tau \otimes \pi)\rho$ which is equivalent to saying that τ is a comodule morphism, since the \mathcal{H} -coaction on $\mathcal{T}(V)$ is given by $(\text{id} \otimes \pi)\rho$. Thus Proposition 4.11 applies and we have a description of $L(A, \mathcal{H})$ as a quotient of $\mathcal{T}(V)$.

(2) Plus, notice that such an A is trivially a cleft object of $L = \mathcal{T}(V)$ and thus $\overline{A} \in \text{Cleft}(\mathcal{H})$. In particular, if \mathcal{H}' is a quotient Hopf algebra of \mathcal{H} , $X' = {}^{\text{co}\mathcal{H}'}\mathcal{H}$ and $f' \in \text{Alg}_L^L(X', \overline{A})$, the corresponding Galois object $B = \overline{A}/\overline{A}f'(X')^+$ is also cleft.

4.2. The graded Hopf algebra associated to a cocycle deformation.

Let H, V be as in (1.1), (1.2) and let \mathcal{B} be a pre-Nichols algebra over V . Set $\mathcal{H} = \mathcal{B}\#H$ and let $\sigma : \mathcal{H} \otimes \mathcal{H} \rightarrow \mathbb{k}$ be a 2-cocycle. Let $\mathfrak{F} = (F_n)_{n \geq 0}$ be the filtration of \mathcal{H}_σ induced by the graduation of \mathcal{H} . Then $\text{gr}_{\mathfrak{F}}\mathcal{H}_\sigma = \mathcal{H}_\sigma = \mathcal{H}$ as coalgebras. Notice that, if \mathcal{B} is a Nichols algebra, then \mathfrak{F} coincides with the coradical filtration.

Proposition 4.13. (a) *There is an isomorphism of graded Hopf algebras $\text{gr}_{\mathfrak{F}}\mathcal{H}_\sigma \simeq \mathcal{B}' \# H_\sigma$, for \mathcal{B}' a pre-Nichols algebra over $V' \in {}^H_\sigma \mathcal{YD}$. Here V' is the H -comodule V with action*

$$(4.4) \quad x \rightharpoonup_\sigma v = \sigma(x_{(1)}, v_{(-2)}) \sigma(x_{(2)} v_{(-1)}, \mathcal{S}(x_{(7)})) \\ \times \sigma^{-1}(x_{(4)}, \mathcal{S}(x_{(5)})) x_{(3)} v_{(0)} \mathcal{S}(x_{(6)})$$

for all $x \in H_\sigma$, $v \in V$ and the product in \mathcal{B}' is given by $x \cdot y = \sigma(x_{(-1)}, y_{(-1)}) x_{(0)} y_{(0)}$, for x, y homogeneous.

(b) *Let $\mathcal{A} \in \text{Cleft}(\mathcal{H})$ with section $\gamma : \mathcal{H} \rightarrow \mathcal{A}$ and consider the induced cocycle $\sigma(x \otimes y) = \gamma(x_{(1)}) \gamma(y_{(1)}) \gamma^{-1}(x_{(2)} y_{(2)})$, $x, y \in \mathcal{H}$, see (2.5). Assume $\gamma|_H \in \text{Alg}(H, \mathcal{A})$. Then $\text{gr}_{\mathfrak{F}}\mathcal{H}_\sigma \simeq \mathcal{B} \# H$.*

(c) *Suppose that $H = \mathbb{k}G$, G a finite group. In particular, $H_\sigma = H$. Let $\{x_1, \dots, x_\theta\}$ be a basis of V with $x_i \in V^{g_i}$, $g_i \in G$, $1 \leq i \leq \theta$. If*

$$(4.5) \quad \sigma(g, g^{-1}) = \sigma(g, g_i) \sigma(g g_i, g^{-1}), \quad g \in G, \quad 1 \leq i \leq \theta,$$

then $V' = V \in {}^H_\sigma \mathcal{YD}$.

Proof. (a) Clearly $\text{gr}_{\mathfrak{F}}\mathcal{H}$ is generated by $H_\sigma \oplus (F_1/H_\sigma)$. Then $\text{gr}_{\mathfrak{F}}\mathcal{H}_\sigma \simeq \mathcal{B}' \# H_\sigma$, where \mathcal{B}' is a pre-Nichols algebra over $V' := \mathcal{B}'^1$. Since the comultiplication is unchanged, $V' = V$ as H_σ -comodules and in $\text{gr}_{\mathfrak{F}}\mathcal{H}_\sigma$:

$$x \rightharpoonup_\sigma v = x_{(1)} \cdot_\sigma v \cdot_\sigma \mathcal{S}(x_{(2)}) = \sigma(x_{(1)} \otimes v_{(-1)}) (x_{(2)} v_{(0)}) \cdot_\sigma \mathcal{S}(x_{(3)}) \\ = \sigma(x_{(1)} \otimes v_{(-2)}) \sigma(x_{(2)} v_{(-1)} \otimes \mathcal{S}(x_{(7)})) \\ \times \sigma^{-1}(x_{(4)} \otimes \mathcal{S}(x_{(5)})) x_{(3)} v_{(0)} \mathcal{S}(x_{(6)})$$

for all $x \in H_\sigma$, $v \in V'$; note that $\Delta(v) = v \otimes 1 + v_{(-1)} \otimes v_{(0)}$. Finally, if $x \in \mathcal{B}^n$ and $y \in \mathcal{B}^m$, then $x \cdot_\sigma y = \sigma(x_{(-1)}, y_{(-1)}) x_{(0)} y_{(0)}$ plus terms of degree lesser than $m + n$. (b) follows since $\sigma|_{H \otimes H} = \varepsilon \otimes \varepsilon$. For (c), the cocommutativity implies $H_\sigma = H$ and plugging (4.5) into (4.4), we have $V' = V$. \square

Let $\mathcal{B} \twoheadrightarrow \mathcal{B}'$ be a quotient of pre-Nichols algebras over $V \in {}^H_\sigma \mathcal{YD}$. Let $L = \mathcal{B} \# H$, $K = \mathcal{B}' \# H$. In particular, L is K -coflat, see Corollary 3.7. Let $A \in \text{Cleft}(L)$ and let $A' \in \text{Cleft}(K)$ be as in Theorem 3.1. Next lemma, which is basically [Sc, Theorem 4.2] in this setting, deals with the context in Proposition 4.13 (b).

Lemma 4.14. *Assume there is a section $\gamma : L \rightarrow A$ with $\gamma|_H \in \text{Alg}(H, A)$. Then there exists a section $\gamma' : K \rightarrow A'$ with $\gamma'|_H \in \text{Alg}(H, A')$.*

Proof. Notice that A is an injective K -comodule, since L is coflat over K and A is L -coflat. Thus, as $\gamma|_H : H \rightarrow A$ is K -colinear, there exists a K -colinear map $\omega : K \rightarrow A$ such that $\omega|_H = \gamma|_H$. Also, ω is convolution-invertible, since its restriction to H is, by [T1, Lemma 14]. Let $\tau : A \rightarrow A'$ be the K -comodule algebra projection. In particular, $\tau\omega|_H \in \text{Alg}(H, A')$. Set $\gamma' = \tau\omega : K \rightarrow A'$. Then γ' is invertible K -colinear and thus induces an isomorphism of K -comodules $A' \simeq A'^{\text{co}K} \otimes K = K$. \square

5. THE STRATEGY FOR COMPUTING ALL COCYCLE DEFORMATIONS

Let H, V be as in (1.1), (1.2). We explain how to compute all liftings L which are cocycle deformations of $\mathcal{B}(V)\#H$. We fix a minimal set \mathcal{G} of homogeneous generators of $\mathcal{J}(V)$; \mathcal{G} is finite by assumption.

5.1. Preliminaries. Assume L is such a lifting, say $L = (\mathcal{B}(V)\#H)_\sigma$. Let A be a $\mathcal{B}(V)\#H$ -Galois object such that $L \simeq L(A, \mathcal{B}(V)\#H)$.

On the one hand, one has $\text{gr}L = \mathcal{B}(V)_\sigma\#H_\sigma$, $V' \in {}^{H_\sigma}_{H_\sigma}\mathcal{YD}$ as in Proposition 4.13. Thus a sufficient condition is that σ restricts to $\varepsilon \otimes \varepsilon$ on $H \otimes H$. On the other hand, this is achieved if the object A is cleft and a section $\gamma : \mathcal{B}(V)\#H \rightarrow A$ satisfies $\gamma|_H \in \text{Alg}(H, A)$, by Proposition 4.13 (b).

5.2. Adapted stratification. We say that a decomposition of \mathcal{G} as a disjoint union $\mathcal{G} = \mathcal{G}_0 \cup \mathcal{G}_1 \cup \dots \cup \mathcal{G}_N$ is an *adapted stratification* when it satisfies the following property. For $0 \leq k \leq N$, we set

$$\begin{aligned} \mathcal{B}^0 &:= T(V), & \mathcal{H}^0 &= T(V)\#H, \\ \mathcal{B}^k &:= T(V)/\langle \mathcal{G}_0 \cup \mathcal{G}_1 \cup \dots \cup \mathcal{G}_{k-1} \rangle, & \mathcal{H}^k &= \mathcal{B}^k\#H. \end{aligned}$$

If $k < N$, then the set \mathcal{G}_k identifies with its image in \mathcal{B}^k , by minimality of \mathcal{G} . We assume that \mathcal{G}_k is a basis of a Yetter-Drinfeld submodule of $\mathcal{P}(\mathcal{B}^k)$; then $\langle \mathcal{G}_k \rangle$ is a Hopf ideal of \mathcal{B}^k , $\langle \mathcal{G}_k \rangle\#H$ is a Hopf ideal of \mathcal{H}^k and $\mathcal{H}^{k+1} \simeq \mathcal{H}^k/\langle \mathcal{G}_k \rangle\#H$ is a Hopf algebra. We also assume that the elements of \mathcal{G}_N form a basis of a Yetter-Drinfeld submodule of \mathcal{H}^N , but they are not necessarily primitive. Notice that $\mathcal{H}^{N+1} = \mathcal{B}(V)\#H$.

Examples 5.1. (a) A standard choice is to take \mathcal{G}_k such that (the image of) \mathcal{G}_k is a basis of $\mathcal{P}(\mathcal{B}^k) \setminus V$ for all k .
 (b) Assume that $H = \mathbb{k}\Gamma$, Γ a finite abelian group, or more generally that V is a direct sum of one-dimensional Yetter-Drinfeld modules. We may choose an adapted stratification with $\text{card } \mathcal{G}_j = 1$ for all j .

Definition 5.2. Let $0 \leq n \leq N$, $\mathcal{A}^n \in \text{Gal}(\mathcal{H}^n)$. Set $X_n = {}^{\mathcal{H}^{n+1}}\mathcal{H}^n$. We say that \mathcal{A}^n is *admissible* if there exist $f_n \in \text{Alg}_{\mathcal{H}^n}^{\mathcal{H}^n}(X_n, \mathcal{A}^n)$.

Let $0 \leq n \leq N$. If $\mathcal{A}^n \in \text{Gal}(\mathcal{H}^n)$ is admissible, then we can apply Theorem 3.1 to \mathcal{H}^n and \mathcal{A}^n to find an object $\mathcal{A}^{n+1} \in \text{Gal}(\mathcal{H}^{n+1})$. In Subsection 5.4 we will study the algebra X_n and the possible maps f_n .

5.3. The strategy. Set $\mathcal{T}(V) = T(V)\#H$.

(1) We start with $\mathcal{H}^0 = \mathcal{T}(V)$ and the Hopf Galois object $\mathcal{A}^0 = \mathcal{T}(V)$ with section $\gamma^0 = \text{id}$. As a result, we have:

- Every quotient object \mathcal{A}^k is cleft, see Remark 4.12 (2).
- For each k , we may assume that the section $\gamma^k : \mathcal{H}^k \rightarrow \mathcal{A}^k$ is such that $\gamma|_H \in \text{Alg}(H, \mathcal{A}^k)$, see Lemma 4.14. In particular, $\gamma^k(H) = H$.

Set $\Lambda^0 = \{\mathcal{T}(V)\}$, let Λ^n be the set of admissible objects of \mathcal{H}^n , $n \geq 1$.

(2) At each step, we use Günther's theorems 3.1 and 3.2 to construct objects $\mathcal{A}^{k+1} \in \text{Gal}(\mathcal{H}^{k+1})$ as quotients of objects $\mathcal{A}^k \in \Lambda^k$.

(3) For each $\mathcal{A}^{N+1} \in \Lambda^{N+1}$, consider $L^{N+1} = L(\mathcal{A}^{N+1}, \mathcal{H}^{N+1})$. We have $\text{gr} L^{N+1} \simeq \mathcal{B}(V) \# H$.

(4) Given a lifting candidate L , we use Theorem 4.9 or Propositions 4.11 and 5.7 below to check if $L \simeq L^{N+1}$ for one of these algebras.

Remark 5.3. A similar strategy is already proposed by Günther in [Gu, Page 4399], to compute the cleft objects of a pointed Hopf algebra \overline{H} which is a quotient of a pointed Hopf algebra H for which $\text{Cleft}(H)$ is known. He suggests to *choose an easy decomposition* $H = H^1 \twoheadrightarrow H^2 \twoheadrightarrow \dots \twoheadrightarrow H^n = \overline{H}$ in such a way that $\text{Cleft}(H^{i+1})$ is *easily computable* from $\text{Cleft}(H_i)$ using Günther's Theorems 3.1 and 3.2. He does not, however, investigate how to find that decomposition or when the method applies, nor relates this process with the lifting procedure or the classification problem.

5.4. A special case. In this part we restrict ourselves to those steps in the stratification in which \mathcal{G}_j is such that $\text{card } \mathcal{G}_j = 1$. More precisely, let $R \in {}^H_H\mathcal{YD}$ be a pre-Nichols algebra over $V \in {}^H_H\mathcal{YD}$. Set $\mathcal{H} = R \# H$ and let $A \in \text{Cleft}(\mathcal{H})$. We assume for the rest of the section that, for some $g \in G(H)$

$$(5.1) \quad \begin{aligned} &\text{there is } u \in \mathcal{P}_{(1,g)}(\mathcal{H}) \quad \text{and} \\ &\text{there is a section } \gamma : \mathcal{H} \rightarrow A, \quad \text{with } \gamma|_H \in \text{Alg}(H, A). \end{aligned}$$

Also, we set

$$(5.2) \quad v = ug^{-1}, \quad \mathcal{H}' = \mathcal{H}/\langle u \rangle, \quad X = {}^{\text{co } \mathcal{H}'} \mathcal{H}.$$

Notice that $v \in X$. Also, $v \in \mathcal{P}_{(g^{-1},1)}(\mathcal{H})$ and if $u \in \mathcal{P}_{(1,g)}^\chi(\mathcal{H})$ for some $\chi \in \widehat{G}(H)$, then g is central in $G(H)$ and $v \in \mathcal{P}_{(g^{-1},1)}^\chi(\mathcal{H})$.

Recall that if there exists $f \in \text{Alg}_{\mathcal{H}}^{\mathcal{H}}(X, A)$, then by Theorem 3.1, we can find $A' \in \text{Cleft}(\mathcal{H}')$ from the data (A, f) . The same is valid if there is $f \in \text{Alg}^{\mathcal{H}}(Y, A)$, for $Y = \mathbb{k}\langle v \rangle \subset X$ the subalgebra generated by v and if $A/Af(v)A \neq 0$. Thus, we shall:

- (i) Describe X in a simple case which, however, presents itself in the examples, see Lemma 5.4.
- (ii) Compute the value of $f(v)$ for any f as above, see Proposition 5.5.
- (iii) Study the shape of the corresponding Hopf algebra $L(A', \mathcal{H}')$ as a quotient of $L(A, \mathcal{H})$, see Proposition 5.7.

Let $\mathbb{k}[T]$ denote the polynomial algebra, $\pi : \mathcal{T}(V) \twoheadrightarrow \mathcal{H}$ the projection.

Lemma 5.4. *If $g^{-1}vg = v$, then there exists an algebra monomorphism $\mathbb{k}[T] \rightarrow X$, given by $T \mapsto v$. If $\mathcal{S}(h_{(1)})vh_{(2)} \in \mathbb{k}\langle v \rangle \forall h \in \mathcal{H}$, then $X \simeq \mathbb{k}[T]$.*

Proof. It suffices to show that for every $r > 0$, $v^r \neq 0$. If not, let $r \in \mathbb{N}$ be minimal with $v^r = 0$. Then $0 = \Delta(v^r) = \sum_{j=1}^{r-1} \binom{r}{j} v^j g^{j-r} \otimes v^{r-j}$, which contradicts the minimality of r . If $\mathcal{S}(h_{(1)})vh_{(2)} \in \mathbb{k}\langle v \rangle \forall h \in \mathcal{H}$, then $\mathbb{k}\langle v \rangle$ is a normal subalgebra of \mathcal{H} , hence the lemma follows by [T2, Theorem 3.2]. \square

Proposition 5.5. *If $f \in \text{Alg}^{\mathcal{H}}(X, A)$, then there exists $c \in \mathbb{k}$ such that*

$$(5.3) \quad f(v) = \gamma(v) - c\gamma(g^{-1}).$$

- (a) *If $v \in \mathcal{P}_{(g^{-1}, 1)}^{\chi}(\mathcal{H})$, $\chi \neq \varepsilon$ and $c \neq 0$, then $\langle f(v) \rangle = A$.*
- (b) *If $f \in \text{Alg}_{\mathcal{H}}^{\mathcal{H}}(X, A)$, then $f(\mathcal{S}(\bar{x}_{(1)})v\bar{x}_{(2)}) = \tau(\mathcal{S}(x_{(1)}))f(v)\tau(x_{(2)})$, for $\bar{x} = \pi(x) \in \mathcal{H}$. If $v \in \mathcal{P}_{(g^{-1}, 1)}^{\chi}(\mathcal{H})$, $\chi \neq \varepsilon$, then $f(v) = \gamma(v)$.*

Proof. As f and γ are \mathcal{H} -colinear,

$$\rho\left((\gamma(v) - f(v))\gamma(g)\right) = (\gamma(v) - f(v))\gamma(g) \otimes 1.$$

Thus, $(\gamma(v) - f(v))\gamma(g) \in A^{\text{co}\mathcal{H}} \simeq \mathbb{k}$ and there is $c \in \mathbb{k}$ such that (5.3) holds. Consider A as a $G(H)$ -module with the adjoint action via γ . If $\chi \neq \varepsilon$, then there exists $t \in G(H)$ such that $\chi(t) \neq 1$. Then

$$tf(v)t^{-1} - \chi(t)f(v) = (\chi(t) - 1)c\gamma(g^{-1}) \in \langle f(v) \rangle.$$

As $\gamma(g^{-1}) \in A$ is invertible, (a) follows. The first assertion of (b) follows since $\text{can}^{-1}(1 \otimes \pi(x)) = \tau(\mathcal{S}(x_{(1)})) \otimes \tau(x_{(2)})$. Indeed,

$$\begin{aligned} \text{can}(\tau(\mathcal{S}(x_{(1)})) \otimes \tau(x_{(2)})) &= \tau(\mathcal{S}(x_{(1)}))\tau(x_{(2)})_{(0)} \otimes \tau(x_{(2)})_{(1)} \\ &= \tau(\mathcal{S}(x_{(1)}))\tau(x_{(2)}) \otimes \pi(x_{(3)}) = 1 \otimes \pi(x). \end{aligned}$$

Set $\xi = \chi^{-1}$, then $f(v) \cdot k = f(v \cdot k) = \xi(k)f(v)$ and $\gamma(v) \cdot k = \xi(k)\gamma(v)$ for $k \in G(H)$. Thus $c\gamma(g^{-1}) = \gamma(v) - f(v) \in A^{\xi} \cap A^{\varepsilon}$. \square

Remark 5.6. Let $G = G(H)$. We shall identify $\mathbb{k}G$ with the subalgebra of A generated by $\gamma(g)$, $g \in G$, and set $g := \gamma(g)$. Also, $\mathbb{k}G$ identifies with the subalgebra of $L(A, \mathcal{H})$ generated by $g \otimes g^{-1}$, $g \in G$, by Proposition 4.11.

Let L be a Hopf algebra such that A is (L, \mathcal{H}) -biGalois object. Fix an algebra map $f : X \rightarrow A$ as in Proposition 5.5 and let $c \in \mathbb{k}$ be as in (5.3). Let $A' = A/\langle f(v) \rangle = A/\langle \gamma(u) - c \rangle$.

Proposition 5.7. (a) *The element*

$$(5.4) \quad (\gamma \otimes \gamma^{-1})\Delta(u) = \gamma(u) \otimes 1 - g \otimes g^{-1}\gamma(u)$$

is $(1, g)$ -primitive in $L(A, \mathcal{H})$.

(b) *Assume $A' \neq 0$. Then $L(A', \mathcal{H}') \simeq L/\langle \tilde{u} - c(1 - g) \rangle$, where \tilde{u} satisfies*

$$(5.5) \quad \tilde{u} \otimes 1 = \gamma(u)_{(-1)} \otimes \gamma(u)_{(0)} - g \otimes \gamma(u).$$

Proof. Let $\rho : A \rightarrow A \otimes H$ be the coaction and denote by $\rho^{(2)}$ the diagonal coaction on $A \otimes A^{\text{op}}$. Let $\bar{u} = (\gamma \otimes \gamma^{-1})\Delta(u)$. It is easy to see that $\rho^{(2)}(\bar{u}) = \bar{u} \otimes 1$ and thus $\bar{u} \in L(A, \mathcal{H})$. Recall the formula for the comultiplication in $L(A, \mathcal{H})$ from (2.7). Then $\Delta(\bar{u}) = \bar{u} \otimes 1 \otimes 1 + g \otimes g^{-1} \otimes \bar{u}$ and (a) follows via the identification in Remark 5.6. Now, (b) follows using formula (2.9). \square

It is not always possible to choose a stratification $\mathcal{G} = \mathcal{G}_0 \cup \mathcal{G}_1 \cup \dots \cup \mathcal{G}_N$ in which $\text{card } \mathcal{G}_j = 1$ for each j . For instance, if G is a finite non-abelian group, then examples of this situation arise when $H = \mathbb{k}G$ or the function algebra \mathbb{k}^G , see Example 5.8 below. In these cases, there are simple Yetter-Drinfeld modules of dimension bigger than one.

Example 5.8. Keep the notation in Example 4.10. The stratification necessary to compute the lifting of $\mathcal{B}(\mathbb{F}_5, 2)$ is: $\mathcal{G}_0 = \{x_i^2 : i \in \mathbb{F}_5\}$, $\mathcal{G}_1 = \{x_i x_j + x_{2j-i} x_i + x_{3i-2j} x_{2j-i} + x_j x_{3i-2j} : i, j \in \mathbb{F}_5\}$ and $\mathcal{G}_2 = \{x_1 x_0 x_1 x_0 + x_0 x_1 x_0 x_1\}$ [GIV, Theorem 5.4]. Note that the Yetter-Drinfeld module generated by \mathcal{G}_k is simple in the pre-Nichols algebra \mathcal{B}^k , $0 \leq k \leq 2$.

As in Example 5.8, the stratifications we will use contain in their first components relations of type $x_i^{N_i}$, as in Example 5.8. In view of Proposition 5.5 we need to study the quotients of $\mathcal{T}(V)$ by relations $x_i^{N_i} - \lambda_i^{N_i}$.

Lemma 5.9. *Let $H = \mathbb{k}\Gamma$, Γ a finite group. Assume that $V \in {}^H_H \mathcal{YD}$ is such that there exist a basis x_1, \dots, x_θ of V , $g_1, \dots, g_\theta \in \Gamma$ and $\chi_1, \dots, \chi_\theta \in \widehat{\Gamma}$ such that $x_i \in V_{\chi_i}^{g_i}$. Fix $\lambda_i \in \mathbb{k}$ such that $\lambda_i = 0$ if $\chi_i^{N_i} \neq \varepsilon$, where N_i is the order of $\chi_i(g_i)$. Then $A := \mathcal{T}(V) / \langle x_i^{N_i} - \lambda_i^{N_i} : i = 1, \dots, \theta \rangle \neq 0$.*

Proof. Note that the order of χ_i is greater or equal than N_i . Up to reordering the basis we can assume that $\chi_i^{N_i} = \varepsilon$ if $1 \leq i \leq k$, and $\chi_i^{N_i} \neq \varepsilon$ if $k < i \leq \theta$. We prove the statement by induction in k . For $r, M \in \mathbb{N}$, we set:

$$P_{r,M} := \begin{pmatrix} 0 & \text{id}_r & 0 & \dots & 0 \\ 0 & 0 & \text{id}_r & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & \text{id}_r \\ \text{id}_r & 0 & 0 & \dots & 0 \end{pmatrix} \in M_{rM}(\mathbb{k}).$$

Also, $D(a_1, \dots, a_M)$ is the block diagonal matrix with entries $a_i \in M_r(\mathbb{k})$, $1 \leq i \leq M$. Now, if $k = 0$, A is a \mathbb{N}_0^θ -graded algebra, because we quotient by homogeneous elements. Let $k = 1$. Let $\bar{\rho} : \mathcal{T}(V) \rightarrow \text{End}(\mathbb{k}^{N_1})$ be given by $g \mapsto D(1, \chi_1^{N_1-1}(g), \dots, \chi_1(g))$, $x_i \mapsto 0$ if $i > 1$, $x_1 \mapsto \lambda_1 P_{1,N_1}$. Note that $\bar{\rho}(g)\bar{\rho}(x_i) = \chi_i(g)\bar{\rho}(x_i)\bar{\rho}(g)$, $i = 1, \dots, \theta$, so it is well-defined. But it also induces a morphism $\rho : A \rightarrow \text{End}(\mathbb{k}^{N_1})$ since

$$\bar{\rho}(x_1^{N_1}) = \lambda_1^{N_1} \text{id} = \bar{\rho}(\lambda_1^{N_1} 1) \quad \text{and} \quad \bar{\rho}(x_i^{N_i}) = 0, \quad i > 1.$$

Suppose that it holds for $k-1 \geq 1$: we can assume that there exists a representation $\tilde{\rho} : \mathcal{T}(\tilde{V}) \rightarrow \text{End}(\mathbb{k}^{N'})$, where \tilde{V} is the subspace of V generated by $x_1, \dots, x_{k-1}, x_{k+1}, \dots, x_\theta$, and $N' = N_1 \cdots N_{k-1}$, satisfying:

$$\begin{aligned} \tilde{\rho}(g)\tilde{\rho}(x_i) &= \chi_i(g)\tilde{\rho}(x_i)\tilde{\rho}(g), \quad i = 1, \dots, \theta+1, \quad i \neq k, \\ \tilde{\rho}(x_i^{N_i}) &= \tilde{\rho}(\lambda_i^{N_i} 1), \quad i < k, \quad \text{and} \quad \tilde{\rho}(x_i^{N_i}) = 0, \quad i > k. \end{aligned}$$

We will obtain now a representation $\bar{\rho} : \mathcal{T}(V) \rightarrow \text{End}(\mathbb{k}^N)$, $N = N'N_k$, obeying the corresponding assumptions. It is defined as follows:

$$\begin{aligned} g &\mapsto D\left(\tilde{\rho}(g), \tilde{\rho}(g)\chi_k^{N_k-1}(g), \dots, \tilde{\rho}(g)\chi_k(g)\right), & g \in \Gamma, \\ x_i &\mapsto D\left(\tilde{\rho}(x_i), \tilde{\rho}(x_i), \dots, \tilde{\rho}(x_i)\right), & i \neq k, \\ x_k &\mapsto \lambda_k P_{N', N_k}. \end{aligned}$$

It follows that $\bar{\rho}|_\Gamma$ is a representation of Γ , because it is a direct sum of representations of Γ . When $i \neq k$ the equations

$$\begin{aligned} \bar{\rho}(g)\bar{\rho}(x_i) &= \chi_i(g)\bar{\rho}(x_i)\bar{\rho}(g), \quad i = 1, \dots, \theta + 1, \quad i \neq k, \\ \bar{\rho}\left(x_i^{N_i}\right) &= \bar{\rho}\left(\lambda_i^{N_i} 1\right), \quad i < k, \quad \text{and} \quad \bar{\rho}(x_i^{N_i}) = 0, \quad i > k, \end{aligned}$$

follow because they hold for $\tilde{\rho}$. Finally we prove that the equations

$$\bar{\rho}(g)\bar{\rho}(x_k) = \chi_k(g)\bar{\rho}(x_k)\bar{\rho}(g), \quad \bar{\rho}\left(x_k^{N_k}\right) = \bar{\rho}\left(\lambda_k^{N_k} 1\right)$$

hold by a proof analogous to the case $k = 1$. Therefore $A \neq 0$, since $\tilde{\rho}$ induces a nonzero morphism $\rho : A \rightarrow \text{End}(\mathbb{k}^N)$. \square

5.5. An example of diagonal type. We apply our strategy to classify the liftings of the Nichols algebra associated to the diagram

$$(5.6) \quad \circ_{-\zeta} \xrightarrow{\zeta^7} \circ_{\zeta^3}, \quad \zeta \in \mathbb{G}_9$$

of [H, Table 1, row 9]. Consider a matrix $(q_{ij})_{1 \leq i, j \leq 2}$ corresponding to (5.6) and let Γ be a finite group such that there is a realization of this braiding, that is there are $g_1, g_2 \in \Gamma$, $\chi_1, \chi_2 \in \hat{\Gamma}$ such that $\chi_j(g_i) = q_{ij}$, $1 \leq i, j \leq 2$. Let $V \in \frac{\mathbb{k}\Gamma}{\mathbb{k}\Gamma} \mathcal{YD}$ be the corresponding Yetter-Drinfeld module, with basis $\{x_1, x_2\}$. By [An, Example 2.5], $\mathcal{B}(V)$ has a presentation by generators x_1, x_2 and relations

$$x_1^{18} = x_2^3 = x_{12}^{18} = [x_1, x_{122}]_c - a x_{12}^2 = 0,$$

where $x_{12} = (\text{ad}_c x_1)x_2$, $x_{122} = [x_{12}, x_2]_c$ and $a = \zeta^7 q_{12}(1 + \zeta)^{-1}$. We fix the following stratification:

$$(5.7) \quad \mathcal{G}_0 = \{x_1^{18}, x_2^3\}, \quad \mathcal{G}_1 = \{[x_1, x_{122}]_c - a x_{12}^2\}, \quad \mathcal{G}_2 = \{x_{12}^{18}\}.$$

Set $\mathcal{H} = \mathcal{B}(V) \# \mathbb{k}\Gamma$ and $\mathcal{H}' = \mathcal{B}'(V) \# \mathbb{k}\Gamma$ where $\mathcal{B}'(V)$ is the pre-Nichols algebra defined by the relations in $\mathcal{G}_0 \cup \mathcal{G}_1$. Let $\lambda_1, \lambda_2, \lambda_3 \in \mathbb{k}$ subject to:

$$\lambda_1 = 0 \quad \text{if } \chi_1^{18} \neq \varepsilon, \quad \lambda_2 = 0 \quad \text{if } \chi_2^3 \neq \varepsilon, \quad \lambda_3 = 0 \quad \text{if } \chi_1^{18} \chi_2^{18} \neq \varepsilon.$$

Let $\mathcal{A}' = \mathcal{A}(\lambda_1, \lambda_2)$ be the quotient of $T(V) \# \mathbb{k}\Gamma$ by the ideal generated by

$$x_1^{18} - \lambda_1, \quad x_2^3 - \lambda_2, \quad [x_1, x_{122}]_c - a x_{12}^2.$$

Let $\mathcal{L}' = \mathcal{L}(\lambda_1, \lambda_2)$ be the quotient of $T(V) \# \mathbb{k}\Gamma$ by the ideal generated by

$$x_1^{18} - \lambda_1(1 - g_1^{18}), \quad x_2^3 - \lambda_2(1 - g_2^3), \quad [x_1, x_{122}]_c - a x_{12}^2.$$

\mathcal{L}' is a Hopf algebra since this is a Hopf ideal.

- Proposition 5.10.** (a) \mathcal{A}' is a right \mathcal{H}' -Galois object with coaction induced by the comultiplication in $\mathcal{T}(V)$.
 (b) $L(\mathcal{A}', \mathcal{H}') = \mathcal{L}'$.
 (c) There exist $a_{12} = a_{12}(\lambda_1, \lambda_2) \in \mathcal{A}'$ such that

$$\mathcal{A}(\lambda_1, \lambda_2, \lambda_3) = \mathcal{A}' / \langle x_{12}^{18} - a_{12} - \lambda_3 \rangle$$

is a right \mathcal{H} -Galois object.

Proof. (a) follows using the Strategy combined with Theorem 3.2 with the stratification (5.7). The algebra obtained in the first step from \mathcal{G}_0 is nonzero by Lemma 5.9. Then we check with [GAP] together with the package [GBNP] that $\mathcal{A}' \neq 0$. Let $\gamma^1 : \mathcal{H}^1 \rightarrow \mathcal{A}^1$ as in 5.3. Notice that

$$\gamma^1([x_1, x_{122}]_c - a x_{12}^2) = [x_1, x_{122}]_c - a x_{12}^2.$$

(b) follows from Proposition 5.7. As for (c), let $\gamma^2 : \mathcal{H}' \rightarrow \mathcal{A}'$. Then a_{12} is defined by $\gamma^2(x_{12}^{18}) = x_{12}^{18} - a_{12}$. We use [GAP] to see that x_{12}^{18} is normal in \mathcal{H}' , see also [AAGI], and (c) follows from Theorem 3.1. \square

As in Proposition 5.7, let $s_{12} = s_{12}(\lambda_1, \lambda_2) \in \mathcal{L}'$ be given by

$$(x_{12}^{18} - s_{12}) \otimes 1 = \gamma^2(x_{12}^{18})_{(-1)} \otimes \gamma^2(x_{12}^{18})_{(0)} - 1 \otimes \gamma^2(x_{12}^{18}).$$

Let $\mathcal{L} = \mathcal{L}(\lambda_1, \lambda_2, \lambda_3)$ be the quotient of \mathcal{L}' by the Hopf ideal generated by

$$x_{12}^{18} - s_{12} - \lambda_3(1 - g_1^{18}g_2^{18}).$$

Theorem 5.11. (a) \mathcal{L} is a cocycle deformation of \mathcal{H} .

(b) $\text{gr}\mathcal{L} \simeq \mathcal{H}$.

(c) Reciprocally, if L is a Hopf algebra such that $\text{gr}L \simeq \mathcal{H}$, then there exist $\lambda_1, \lambda_2, \lambda_3$ such that $L \simeq \mathcal{L}(\lambda_1, \lambda_2, \lambda_3)$.

Proof. (a) It follows from Proposition 5.7 that $\mathcal{L} = L(\mathcal{A}, \mathcal{H})$.

(b) follows by Proposition 4.13.

(c) Let $\phi : \mathcal{T}(V) \rightarrow L$ be as in Definition 4.2. If $r \in \mathcal{G}_0 \cup \mathcal{G}_1$, then r is $(g(r), 1)$ -primitive for $g(r) \in \Gamma$, hence $\phi(r) \in L_1$. Let $\chi_r \in \widehat{\Gamma}$ be the character from the Γ -action on r . Now, the pair $(\chi_r, g(r))$ is different from (χ_i, g_i) , $i = 1, 2$ and thus $\phi(r) \in \mathbb{k}\Gamma$. Indeed,

$$\chi_1^{18}(g_1^{18}) = \chi_2^3(g_2^3) = 1, \quad \chi_1^2\chi_2^2(g_1^2g_2^2) = \zeta^8.$$

Also, $\chi_1^2\chi_2^2 \neq \varepsilon$. Then there exist $\lambda_1, \lambda_2 \in \mathbb{k}$ such that ϕ factorizes through $\mathcal{L}(\lambda_1, \lambda_2)$ by [AS3, Lemma 6.1]. By Proposition 5.7, $x_{12}^{18} - s_{12}$ is $(g_1^{18}g_2^{18}, 1)$ -primitive and thus $\phi(x_{12}^{18} - s_{12}) \in \mathbb{k}\Gamma$ again by [AS3, Lemma 6.1]. Thus there exists $\lambda_3 \in \mathbb{k}$ such that ϕ factorizes through $\mathcal{L}(\lambda_1, \lambda_2, \lambda_3)$ and induces an isomorphism since both algebras have dimension $\dim \mathcal{B}(V)|\Gamma|$. \square

5.6. A question. Let H be a Hopf algebra with bijective antipode and assume that H_0 is a Hopf subalgebra of H . Let $u \in H$ be a $(1, g)$ -primitive element, that is $\Delta(u) = u \otimes 1 + g \otimes u$. Set $v = g^{-1}u$, $Y = \mathbb{k}\langle v \rangle$ the subalgebra generated by v . Notice that $I = HY^+H = HvH$ is a Hopf ideal and let $\bar{H} = H/I$. Assume that H is \bar{H} -coflat.

Let $A \in \text{Cleft}(H)$, to find a cleft object $\bar{A} \in \text{Cleft}(\bar{H})$ we can either apply

- (a) [Gu, Theorem 4], see Theorem 3.1:
 - (i) Compute $X = {}^{\text{co}\bar{H}}H$ and
 - (ii) find $f \in \text{Alg}_{\bar{H}}^H(X, A)$ or
- (b) [Gu, Theorem 8], see Theorem 3.2:
 - (i) Find $f \in \text{Alg}^H(Y, A)$ and
 - (ii) prove that $Af(v)A \neq A$.

If (a) or (b) are fulfilled, then $\bar{A} = A/\langle f(v) \rangle \in \text{Cleft}(\bar{H})$.

Both alternatives present a hard computational obstacle, together with a straightforward step: Indeed, alternative (a) supposes a presentation of the algebra X by generators and relations, which is not clear how to produce (think of $V \in {}_{\mathbb{k}\Gamma}^{\mathbb{k}\Gamma}\mathcal{YD}$, Γ a finite group, such that there is $x \in V^g$, $g \in \Gamma$, and $n \in \mathbb{N}$ for which $x^n \in T(V) \# \mathbb{k}\Gamma$ is primitive). On the bright side, this alternative ensures us that $\bar{A} \neq 0$.

On the other hand, in alternative (b), while it is easy to compute $f \in \text{Alg}^H(Y, A)$, there is no canonical way to see whether $Af(v)A = A$ or not.

Hence we need an *intermediate Günther's Theorem* exploiting the benefits of both alternatives. That being said, we collect from the examples enough evidence to change alternative (b), step (i) by

- (b) (i') Find $f \in \text{Alg}_{H_0}^H(Y^0, A)$, $Y^0 = Y \cdot H_0$.

Here $\text{Alg}_{H_0}^H(-, -)$ denotes the space of algebra morphisms in $\mathcal{YD}_{H_0}^H$, the full subcategory of \mathcal{YD}_H^H composed by H -comodules and H_0 -modules. Actually, in many examples we see that $Y^0 = Y$ (e.g. $H_0 = \mathbb{k}\Gamma$, Γ abelian) and that not only $f \in \text{Alg}_{H_0}^H(Y^0, A)$ induces a nonzero algebra \bar{A} but also that any $\bar{A} \in \text{Cleft}(\bar{H})$ and hence any $F \in \text{Alg}_{\bar{H}}^H(X, A)$ is determined by $f = F|_Y \in \text{Alg}^H(Y, A)$. Furthermore, in general,

- (1) X is the subalgebra generated by $Y \cdot H$, see [T2, Theorem 3.2].

Question 1. Is there a general setting in which any $f \in \text{Alg}_{H_0}^H(Y^0, A)$ extends to $F \in \text{Alg}_{\bar{H}}^H(X, A)$ with $F|_Y = f$?

Assume that H_0 is finite-dimensional. Then evidence of a positive answer is given by (1) above and the fact that

- (2) A is an injective object in $\mathcal{YD}_{H_0}^H$ and hence any $f \in \text{Hom}_{H_0}^H(Y^0, A)$ extends to $F \in \text{Hom}_{H_0}^H(X, A)$ with $F|_Y = f$.

Indeed, $\mathcal{YD}_{H_0}^H \simeq \mathcal{M}^L$, with $L = H \blacktriangleleft_{\tau} H_0^{*\text{cop}}$, see [M, Exercise 7.2.16]. Here $\tau = \sum_i \mathcal{S}(e^i) \otimes e_i \in H_0^{*\text{cop}} \otimes H$, for dual bases $\{e_i\}$, $\{e^i\}$ of H_0 and H_0^* . In particular, $L \simeq H \otimes H_0^{*\text{cop}}$ as algebras and there is a Hopf algebra

projection $L \twoheadrightarrow H$, see *loc. cit.* for details. Then L -coflatness follows from H -coflatness since $L \twoheadrightarrow H$ is a cosemisimple coextension. Recall that A is H -coflat since $A \simeq H$ in \mathcal{M}^H .

As a last word, we may also assume that:

- (I) The coradical of \bar{H} is liftable. Moreover, $(\bar{H})_0 \simeq H_0$, as we deal with $H = R \# H_0$, $R \in {}^{H_0}\mathcal{YD}$ and $\bar{H} = \bar{R} \# H_0$, for $R \twoheadrightarrow \bar{R} \in {}^{H_0}\mathcal{YD}$.
- (II) There is a section $\gamma : H \rightarrow A$ with $\gamma|_{H_0} \in \text{Alg}(H_0, A)$. Then, if $\bar{A} \neq 0$, there is a section $\bar{\gamma} : \bar{H} \rightarrow \bar{A}$ with $\bar{\gamma}|_{H_0} \in \text{Alg}(H_0, \bar{A})$, see Lemma 4.14. This also gives $A \simeq H$ in $\mathcal{YD}_{H_0}^H$.

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